

RANDOM LOCATIONS, ORDERED RANDOM SETS AND STATIONARITY

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ABSTRACT. Intrinsic location functional is a large class of random locations containing locations that one may encounter in many cases, *e.g.*, the location of the path supremum/infimum over a given interval, the first/last hitting time, *etc.* It has been shown that this notion is very closely related to stationary stochastic processes, and can be used to characterize stationarity. In this paper the author firstly identifies a subclass of intrinsic location functional and proves that this subclass has a deep relationship to stationary increment processes. Then we describe intrinsic location functionals using random partially ordered point sets and piecewise linear functions. It is proved that each random location in this class corresponds to the location of the maximal element in a random set over an interval, according to certain partial order. Moreover, the locations changes in a very specific way when the interval of interest shifts along the real line. Based on these ideas, a generalization of intrinsic location functional called "local intrinsic location functional" is introduced and its relationship with intrinsic location functional is investigated.

1. INTRODUCTION

Stationarity has been an essential concept in stochastic processes since very long, both due to its theoretical importance and to its extensive use in modeling. Many related problems, especially extreme values of stationary processes, have attracted intensive and ongoing research interests. The classical text Leadbetter et al. (1983) and the new book Lindgren (2012) are both excellent sources for summaries of existing results and literature reviews. Meanwhile, the random locations of stationary processes, such as the location of the path supremum over an interval or the first hitting time of certain level over an interval, have received relatively less attention, particularly in a general setting, when the process is not from one of the few well studied "nice" classes.

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In the paper Samorodnitsky and Shen (2013b), the authors introduced a new notion called "intrinsic location functional", as an abstraction of the common random locations often considered. More precisely, let H be a space of real valued functions on \mathbb{R} , closed under shift. That is, for any $f \in H$ and $c \in \mathbb{R}$, the function $\theta_c f$, defined by $\theta_c f(x) = f(x + c)$, $x \in \mathbb{R}$ is also in H . Examples of H include the space of all continuous functions $\mathcal{C}(\mathbb{R})$, the space of all càdlàg functions $\mathcal{D}(\mathbb{R})$, the space of all upper semi-continuous functions, *etc.* Equip H with the cylindrical σ -field. Let \mathcal{I} be the set of all compact, non-degenerate intervals in \mathbb{R} : $\mathcal{I} = \{[a, b] : a < b, [a, b] \subset \mathbb{R}\}$.

Definition 1.1. *A mapping $L : H \times \mathcal{I} \rightarrow \mathbb{R} \cup \{\infty\}$ is called an intrinsic location functional, if it satisfies the following conditions.*

- (1) *For every $I \in \mathcal{I}$ the map $L(\cdot, I) : H \rightarrow \mathbb{R} \cup \{\infty\}$ is measurable.*
- (2) *For every $f \in H$ and $I \in \mathcal{I}$, $L(f, I) \in I \cup \{\infty\}$.*
- (3) *(Shift compatibility) For every $f \in H$, $I \in \mathcal{I}$ and $c \in \mathbb{R}$,*

$$L(f, I) = L(\theta_c f, I - c) + c,$$

where $I - c$ is the interval I shifted by $-c$, and $\infty + c = \infty$.

- (4) *(Stability under restrictions) For every $f \in H$ and $I_1, I_2 \in \mathcal{I}$, $I_2 \subseteq I_1$,*

$$\text{if } L(f, I_1) \in I_2, \text{ then } L(f, I_2) = L(f, I_1).$$

- (5) *(Consistency of existence) For every $f \in H$ and $I_1, I_2 \in \mathcal{I}$, $I_2 \subseteq I_1$,*

$$\text{if } L(f, I_2) \neq \infty, \text{ then } L(f, I_1) \neq \infty.$$

It is not difficult to realize that intrinsic location functional is an abstraction of common random locations such as the location of the path supremum/infimum over an interval, the first/last hitting time over an interval, among many others. Interested readers are invited to see Samorodnitsky and Shen (2013b) for more examples and counterexamples of intrinsic location functionals. Notice that in the definition we included ∞ as a possible value. This corresponds to the fact that not all the random locations are necessarily well-defined for all the paths. For instance, a path can lie above certain level over the whole interval of interest, leaving the first/last hitting time undefined. Here and later, we always assign ∞ as the value of an intrinsic

location functional when it is otherwise undefined. Accordingly, the σ -field used for $\mathbb{R} \cup \{\infty\}$ is generated by the Borel σ -field plus ∞ as a singleton.

It turns out that, despite the huge variety of the origins and natures of these random locations, the common points that they share, now summarized in the definition of intrinsic location functional, are sufficient to guarantee many interesting and important properties of their distributions for stationary processes. The majority of these properties are firstly studied in Samorodnitsky and Shen (2012) and Samorodnitsky and Shen (2013a), for the location of path supremum over compact intervals.

Fix a path space H . Let us denote the stochastic process by \mathbf{X} , with all sample paths in H , and the intrinsic location functional by L . Then for each fixed interval $I = [a, b] \in \mathcal{I}$, $L(\mathbf{X}, I)$ is a random variable taking value on $\mathbb{R} \cup \{\infty\}$. Denote its cumulative distribution function by $F_{\mathbf{X}, I}$ or $F_{\mathbf{X}, [a, b]}$. When the stationarity is assumed, it is clear that the location of the interval I will not affect the distribution of $L(\mathbf{X}, I) - a$, as long as the length of the interval, $|I| = b - a$, remains constant. In this case we often fix the starting point a to be 0, and use the shorter notation $F_{\mathbf{X}, b}$.

Theorem 1.2. *[Samorodnitsky and Shen (2013b)] Let L be an intrinsic location functional and $\mathbf{X} = (X(t), t \in \mathbb{R})$ a stationary process. Then the restriction of the law $F_{\mathbf{X}, T}$ to the interior $(0, T)$ of the interval is absolutely continuous. The density, denoted by $f_{\mathbf{X}, T}$, can be taken to be equal to the right derivative of the cdf $F_{\mathbf{X}, T}$, which exists at every point in the interval $(0, T)$. In this case the density is right continuous, has left limits, and has the following properties.*

(a) *The limits*

$$f_{\mathbf{X}, T}(0+) = \lim_{t \rightarrow 0} f_{\mathbf{X}, T}(t) \text{ and } f_{\mathbf{X}, T}(T-) = \lim_{t \rightarrow T} f_{\mathbf{X}, T}(t)$$

exist.

(b) *The density has a universal upper bound given by*

$$(1.1) \quad f_{\mathbf{X}, T}(t) \leq \max\left(\frac{1}{t}, \frac{1}{T-t}\right), \quad 0 < t < T.$$

(c) The density has a bounded variation away from the endpoints of the interval. Furthermore, for every $0 < t_1 < t_2 < T$,

(1.2)

$$TV_{(t_1, t_2)}(f_{\mathbf{X}, T}) \leq \min(f_{\mathbf{X}, T}(t_1), f_{\mathbf{X}, T}(t_1-)) + \min(f_{\mathbf{X}, T}(t_2), f_{\mathbf{X}, T}(t_2-)),$$

where

$$TV_{(t_1, t_2)}(f_{\mathbf{X}, T}) = \sup \sum_{i=1}^{n-1} |f_{\mathbf{X}, T}(s_{i+1}) - f_{\mathbf{X}, T}(s_i)|$$

is the total variation of $f_{\mathbf{X}, T}$ on the interval (t_1, t_2) , and the supremum is taken over all choices of $t_1 < s_1 < \dots < s_n < t_2$.

(d) The density has a bounded positive variation at the left endpoint and a bounded negative variation at the right endpoint. Furthermore, for every $0 < \varepsilon < T$,

$$(1.3) \quad TV_{(0, \varepsilon)}^+(f_{\mathbf{X}, T}) \leq \min(f_{\mathbf{X}, T}(\varepsilon), f_{\mathbf{X}, T}(\varepsilon-))$$

and

$$(1.4) \quad TV_{(T-\varepsilon, T)}^-(f_{\mathbf{X}, T}) \leq \min(f_{\mathbf{X}, T}(T-\varepsilon), f_{\mathbf{X}, T}(T-\varepsilon-)),$$

where for any interval $0 \leq a < b \leq T$,

$$TV_{(a, b)}^\pm(f_{\mathbf{X}, T}) = \sup \sum_{i=1}^{n-1} (f_{\mathbf{X}, T}(s_{i+1}) - f_{\mathbf{X}, T}(s_i))_\pm$$

is the positive (negative) variation of $f_{\mathbf{X}, T}$ on the interval (a, b) , and the supremum is taken over all choices of $a < s_1 < \dots < s_n < b$.

(e) The limit $f_{\mathbf{X}, T}(0+) < \infty$ if and only if $TV_{(0, \varepsilon)}(f_{\mathbf{X}, T}) < \infty$ for some (equivalently, any) $0 < \varepsilon < T$, in which case

$$(1.5) \quad TV_{(0, \varepsilon)}(f_{\mathbf{X}, T}) \leq f_{\mathbf{X}, T}(0+) + \min(f_{\mathbf{X}, T}(\varepsilon), f_{\mathbf{X}, T}(\varepsilon-)).$$

Similarly, $f_{\mathbf{X}, T}(T-) < \infty$ if and only if $TV_{(T-\varepsilon, T)}(f_{\mathbf{X}, T}) < \infty$ for some (equivalently, any) $0 < \varepsilon < T$, in which case

$$(1.6) \quad TV_{(T-\varepsilon, T)}(f_{\mathbf{X}, T}) \leq \min(f_{\mathbf{X}, T}(T-\varepsilon), f_{\mathbf{X}, T}(T-\varepsilon-)) + f_{\mathbf{X}, T}(T-).$$

The key properties in this theorem, (c), (d) and (e), are called "total variation constraints", since they put constraints on the total variation of the density functions. It was then proved that the total variation constraints of the intrinsic location functionals are not merely a group of properties of

stationary processes: they are actually the stationarity itself, viewed from a different angle.

Theorem 1.3. *[Samorodnitsky and Shen (2013b)] Let \mathbf{X} be a stochastic process with continuous sample paths. The following statements are equivalent.*

- (1) *The process \mathbf{X} is stationary.*
- (2) *For some (equivalently, any) $\Delta > 0$, any intrinsic location functional $L : C(\mathbb{R}) \times \mathcal{I} \rightarrow \mathbb{R} \cup \{\infty\}$, the law of $L(\mathbf{X}, I) - a$, $I = [a, a + \Delta] \in \mathcal{I}$, does not depend on a .*
- (3) *For any intrinsic location functional $L : C(\mathbb{R}) \times \mathcal{I} \rightarrow \mathbb{R} \cup \{\infty\}$, any interval $I = [a, b] \in \mathcal{I}$, the law of $L(\mathbf{X}, I)$ is absolutely continuous on (a, b) and has a density satisfying the total variation constraints.*

To sum up, the notion of intrinsic location functional has been introduced, and its deep relationship to the stationarity has been revealed. It can even be used as an alternative definition of stationarity.

On the other hand, there remain very important questions to ask. Firstly, are there similar results for larger families of stochastic processes compared to stationary processes? The set of stationary increment processes, for instance, includes all the stationary processes, but also many commonly used non-stationary processes, such as Brownian motion or Lévy processes in general. What properties do the distributions of random locations of these processes have? Secondly, there has not been many results developed to describe the object of intrinsic location functional, therefore it is also interesting to proceed in this direction. Some representation results, for example, will also be very valuable.

In this paper, we will answer the questions in these two directions. A subclass of intrinsic location functionals, called “doubly intrinsic location functionals”, will be identified, and its deep relation with stationary increment processes will be investigated. For the other direction, we develop equivalent descriptions, as well as an important generalization, of intrinsic location functionals. These new results will be highly helpful for a better and more comprehensive understanding of the notion of intrinsic location functional.

The rest of the paper is organized in the following way. In part two we define the “doubly intrinsic location functionals”, and show that this subclass of intrinsic location functional can be used to fully characterize the stationarity of the increments of a process. In part three, a generalization of intrinsic location functional called "local intrinsic location functional" is introduced, which allows one to define a random location only for intervals with a single fixed length. Then we develop descriptions for it and also for intrinsic location functionals using partially ordered random point sets. The relation between local intrinsic location functional and intrinsic location functional is investigated in part four, showing that the former naturally inherits most of the properties of the latter. We provide yet another description in part five, which focuses on characterizing the value of a (local) intrinsic location functional as a function of the location of the interval of interest when the length of the interval is fixed.

2. RANDOM LOCATIONS OF STATIONARY INCREMENT PROCESSES

Certain intrinsic location functionals, such as the location of the path supremum/infimum over an interval, the hitting times of the derivative of the path assuming it is C^1 , possess the property of “vertical shift invariance”, in the sense that their values will not change when the path is shifted vertically. In order to benefit from this additional property, we add the vertical shift invariance to the definition to form the new notion of “doubly intrinsic location functional”.

Definition 2.1. *An intrinsic location functional L is called doubly intrinsic, if for every function $f \in H$, every interval $I \in \mathcal{I}$ and every $c \in \mathbb{R}$,*

$$L(f, I) = L(f + c, I).$$

Denote by \mathcal{D} the set of all doubly intrinsic location functionals defined on H .

The word “doubly” in the name refers to the fact that L is both “horizontally shift compatible”, in the sense that it moves along with the function and the interval horizontally, and “vertically shift invariant”, in the sense that it does not move along with the function vertically.

In general, once we verify that certain location is an intrinsic location functional, it is very easy to check whether it is doubly intrinsic or not. Intuitively, an intrinsic location functional is doubly intrinsic if and only if its value only depends on the “shape” of the function and does not depend on the “height” of the function. Here are some most natural and important examples of doubly intrinsic location functionals.

Example 2.2. Let H be the space of all the upper (lower) semi-continuous functions. Then the location of the path supremum (infimum) over an interval I ,

$$\tau_{f,I} := \inf\{t \in I : f(t) = \sup(\inf)_{s \in I} f(s)\}$$

is a doubly intrinsic location functional. The infimum outside means that in case of a tie, we always chose the leftmost point among all the points achieving the path supremum (infimum).

Example 2.3. Let H be the space of all càdlàg functions. Then the time of the first jump in a period $[a, b]$,

$$T_{f,[a,b]}^\Delta := \inf\{t \in [a, b], f(t-) \neq f(t)\}$$

is a doubly intrinsic location functional.

Needless to say, any random location which only depends on the value of the first derivative of \mathcal{C}^1 functions is also doubly intrinsic. For instance, the location of the first local maxima, the first time that the derivative hits certain level, *etc.* The class of doubly intrinsic location functionals extends, however, far beyond these “natural” examples. Actually, let H, H' be two spaces of functions, and φ be a mapping from H to H' which is interchangeable with translation:

$$(2.1) \quad \forall f \in H, \forall c \in \mathbb{R}, \varphi(\theta_c f) = \theta_c(\varphi f),$$

and consistent with vertical shift:

$$(2.2) \quad \forall f \in H, \forall c \in \mathbb{R}, \exists c' \in \mathbb{R}, \varphi(f + c) = \varphi(f) + c'.$$

If L' is a doubly intrinsic location functional on $H' \times \mathcal{I}$, then the functional L on $H \times \mathcal{I}$, defined by

$$L(f, I) := L'(\varphi f, I), \quad \forall f \in H, \forall I \in \mathcal{I},$$

is also a doubly intrinsic location functional, provided that the measurability condition is satisfied. We call it the doubly intrinsic location functional induced by φ . This procedure allows us to associate random locations which are originally only well-defined for “nice” functions to the functions which does not possess the required properties. The transforms satisfying (2.1) and (2.2) include many commonly used operations such as convolution with a given function, differentiation, moving average, moving difference, *etc.*

Example 2.4. Let ψ be the classical mollifier:

$$\psi(x) = \begin{cases} e^{-1/(1-|x|^2)} & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1 \end{cases},$$

then the operation of convolution with ψ transforms any measurable function to a smooth function. That is, let f be any measurable function, then $f * \psi$ is a smooth function, where “ $*$ ” denotes convolution. This convolution is obviously interchangeable with translation. It is easy to see that the location of the first hitting time of the derivative to level h over an interval:

$$L'(g, I) := \inf\{t \in I : g'(t) = h\}$$

(following the tradition that $\inf \phi = \infty$) is a doubly intrinsic location functional on the space of all smooth functions.

We will call a set H of functions on \mathbb{R} a *LI set* (from *locally integrable*) if it has following properties:

- H is invariant under shifts;
- H is equipped with its cylindrical σ -field \mathcal{C}_H ;
- the map $H \times \mathbb{R} \rightarrow \mathbb{R}$ defined by $(f, t) \rightarrow f(t)$ is measurable;
- any $f \in H$ is locally integrable.

An example of *LI set* is the space $\mathcal{D}(\mathbb{R})$ of càdlàg functions on \mathbb{R} . Note that, by Fubini’s theorem, for any *LI set* H , the map $T_\psi : H \rightarrow \mathcal{C}(\mathbb{R})$, defined by

$$(2.3) \quad T_\psi(f) = f * \psi = \int_{-\infty}^{\infty} f(s)\psi(t-s)ds, \quad t \in \mathbb{R}$$

is $\mathcal{C}_H/\mathcal{C}_{\mathcal{C}(\mathbb{R})}$ -measurable. Therefore, if, moreover, the space H in this example is a *LI set*, then the measurability issue for the induced location functional

$$L(f, I) := L'(f * \psi, I)$$

is guaranteed. Thus L is also a doubly intrinsic location functional, now defined on any LI set. The doubly intrinsic location functionals of this kind will play an important role in the proof of the theorem below.

Theorem 2.5. *Let \mathbf{X} be a stochastic process having path in H with probability 1, where H is a LI set. Then the followings are equivalent.*

- (1) *The process \mathbf{X} is of stationary increments.*
- (2) *For some (equivalently, any) $\Delta > 0$, any doubly intrinsic location functional $L : H \times \mathcal{I} \rightarrow \mathbb{R} \cup \{\infty\}$, the law of $L(\mathbf{X}, I) - a$, $I = [a, a + \Delta] \in \mathcal{I}$, does not depend on a .*
- (3) *For any doubly intrinsic location functional $L : H \times \mathcal{I} \rightarrow \mathbb{R} \cup \{\infty\}$, any interval $I = [a, b] \in \mathcal{I}$, The law of $L(\mathbf{X}, I)$ is absolutely continuous on (a, b) and has a density satisfying the total variation constraints (1.2)-(1.6) .*

Similar to the case of intrinsic location functionals and stationary processes, this theorem shows that there is a deep and fundamental relationship between the stationarity of increments, the shift invariance of the distributions of doubly intrinsic locations, and the total variation constraints. The most surprising part is that the total variation constraints alone are enough to imply the stationarity of increments, even there is no distributional invariance explicitly formulated at all. Intuitively, it seems to be totally possible that all the doubly intrinsic location functionals always satisfy the total variation constraints, yet their distributions change over different period. This theorem, however, tells us that this will never happen. The total variation constraints automatically lead to the distributional invariance under translation. It could be the case that for some doubly intrinsic location functional, its distribution varies over time while always keeping the total variation constraints obeyed; but then there must be some other doubly intrinsic location functional, for which the total variation constraints are violated. As a family of random locations, the doubly intrinsic location functional is rich enough such that the total variation constraints on this family provide enough information to guarantee the stationarity of the increment of the process.

It is also interesting to make a comparison between Theorem 2.5 and its stationary counterpart, Theorem 1.3. In each of these cases, we have two spaces: the space of processes and the space of location functionals. In Theorem 1.3, the space of processes is the stationary processes, and the corresponding space of location functionals is the intrinsic location functionals. The two spaces are related one to each other via the total variation constraints. In this sense, the total variation constraints introduce a “duality” between the space of processes and the space of random locations. In Theorem 2.5, the space of processes becomes the stationary increment processes. Notice that since stationary processes are automatically of stationary increments but the converse is not true, the space of stationary increment processes is strictly larger than the space of stationary processes. Therefore we should expect a smaller space of the locations on the other side of the duality. It is indeed the case here, since doubly intrinsic location functionals is by definition a proper subset of intrinsic location functionals. In conclusion, Theorem 2.5 and Theorem 1.3 have the same nature, but are with different sizes of the sets on both sides of the duality.

Now let us turn to the proof of Theorem 2.5. The proof actually highly resembles the corresponding proof of Theorem 1.3 presented in Samorodnitsky and Shen (2013b). The full proof will have four directions: $(1) \rightarrow (2)$, $(1) \rightarrow (3)$, $(2) \rightarrow (1)$ and $(3) \rightarrow (1)$. Given the fact that the proofs for some directions are very long, we will not include everything in the proof below, but will refer to the same proofs in Samorodnitsky and Shen (2013b) when it is possible. Many lemmas and settings, however, require changes and reverification.

First of all, notice the following lemma:

Lemma 2.6. *Let \mathbf{X} be a stationary increment process with paths in H almost surely. Let $L \in \mathcal{D}$ and denote by $F_{\mathbf{X},I}(\cdot)$ the distribution of $L(\mathbf{X}, I)$. Then*

(i) *For any $\Delta \in \mathbb{R}$,*

$$F_{\mathbf{X},[\Delta, T+\Delta]}(\cdot) = F_{\mathbf{X},[0, T]}(\cdot - \Delta).$$

(ii) *For any intervals $[c, d] \subseteq [a, b]$,*

$$F_{\mathbf{X},[a, b]}(B) \leq F_{\mathbf{X},[c, d]}(B) \text{ for any Borel set } B \subseteq [c, d].$$

(iii) For any intervals $[c, d] \subseteq [a, b]$,

$$F_{\mathbf{X}, [a, b]}(\{\infty\}) \leq F_{\mathbf{X}, [c, d]}(\{\infty\}).$$

Proof. The point (ii) and (iii) are direct results of the stability under restriction and the consistency of existence in the definition of intrinsic location functionals, respectively. For (i), define process $\mathbf{Y}(t) := \mathbf{X}(t) - \mathbf{X}(\Delta) + \mathbf{X}(0)$, $t \in \mathbb{R}$, then the stationarity of the increments implies that the process $\mathbf{Y}(\cdot + \Delta)$ has the same distribution as $\mathbf{X}(\cdot)$. Thus

$$F_{\mathbf{X}, [\Delta, T+\Delta]}(\cdot) = F_{\mathbf{Y}, [0, T]}(\cdot).$$

Although $\mathbf{Y}(t) - \mathbf{X}(t) = \mathbf{X}(0) - \mathbf{X}(\Delta)$ is random and depends on the realization, it is a constant over time. Thus

$$L(\mathbf{X}, [0, T]) = L(\mathbf{Y}, [0, T]),$$

hence

$$F_{\mathbf{X}, [0, T]}(\cdot) = F_{\mathbf{Y}, [0, T]}(\cdot).$$

□

The rest of the proof in the direction (1) \rightarrow (2) and (1) \rightarrow (3) follows in the same way as in Samorodnitsky and Shen (2013b).

To prove that (2) \rightarrow (1), consider the following location functional:

$$G_{\mathbf{t}, \mathbf{I}}(\mathbf{X}, [a, a + \Delta]) := \inf\{t \in [a, a + \Delta] : t \in S(\mathbf{X}, \mathbf{t}, \mathbf{I})\},$$

where the random set of points S is defined by

$$S(\mathbf{X}, \mathbf{t}, \mathbf{I}) := \{t \in \mathbb{R} : \mathbf{X}(t + t_i) - \mathbf{X}(t) \in I_i, \forall i = 1, \dots, n\},$$

n is a positive integer, $\mathbf{t} = (t_1, \dots, t_n)$ such that $0 < t_1 < \dots < t_n$, and $\mathbf{I} = I_1 \times \dots \times I_n \in \mathcal{I}^n$. It is then easy to check that such defined $G_{\mathbf{t}, \mathbf{I}}$ is a doubly intrinsic location functional for any $n = 1, 2, \dots$, any \mathbf{t} and \mathbf{I} . Moreover, $G_{\mathbf{t}, \mathbf{I}}(\mathbf{X}, [a, a + \Delta]) = a$ if and only if

$$\mathbf{X}(a + t_i) - \mathbf{X}(a) \in I_i, \forall i = 1, \dots, n.$$

If the distribution of $G_{\mathbf{t}, \mathbf{I}}$ does not depend on a , the probability that $\mathbf{X}(a + t_i) - \mathbf{X}(a) \in I_i, \forall i = 1, \dots, n$. can not depend on a . Since this shift invariance holds for all n, \mathbf{t} and \mathbf{I} , the stationarity of the increments is guaranteed.

We are now left with the proof that (3) \rightarrow (1). The main object that we are going to consider are the doubly intrinsic location functionals of the type of Example 2.4, but slightly more complicated. More precisely, let the function ψ as defined in Example 2.4. Define process $\mathbf{Y} := \mathbf{X} * \psi$, then \mathbf{Y} is a stationary increment process with smooth path. Consequently, $\mathbf{Z} = \mathbf{Y}'$, the derivative of \mathbf{Y} , is a smooth stationary process. For any $n = 1, 2, \dots$, any $h > 0$, $d \geq 0$, any $\mathbf{t} = (t_0, t_1, \dots, t_n)$ such that $0 < t_0 < t_1 < \dots < t_n$ and any $\mathbf{I} = I_1 \times \dots \times I_n \in \mathcal{I}^n$, define the random set of points

$$A_{\mathbf{t}, \mathbf{I}}^{h,d}(\mathbf{X}) = \{s \in \mathbb{R} : \mathbf{Z}(s) = h, \inf\{r > s : \mathbf{Z}(r) = h\} > t + d, \\ \mathbf{X}(s + t_i) - \mathbf{X}(s + t_0) \in I_i, \forall i = 1, \dots, n\}.$$

Notice that the LI setting guarantees the measurability. This set seems to be a little strange at the first glance, since the points are marked according to the process \mathbf{Z} , but then filtered using conditions on the original process \mathbf{X} . However, since \mathbf{Z} is transformed from \mathbf{X} and both the operation of convolution and differentiation are interchangeable with translation, the location

$$L(\mathbf{X}, I) := \inf\{t : t \in A_{\mathbf{t}, \mathbf{I}}^{h,d}(\mathbf{X}) \cap I\}$$

is an intrinsic location functional. Moreover, since the points are marked on the derivative \mathbf{Z} and then filtered using conditions only on the increments $\mathbf{X}(s + t_i) - \mathbf{X}(s + t_0)$, the location $L(\mathbf{X}, I)$ is invariant under vertical shift. Hence $L(\mathbf{X}, I)$ is a doubly intrinsic location functional. After defining

$$(2.4) \quad p_{\mathbf{t}, \mathbf{I}, a, \Delta}^{h,d}(\mathbf{X}) = \mathbb{P}(A_{\mathbf{t}, \mathbf{I}}^{h,d}(\mathbf{X}) \cap [a, a + \Delta] \neq \emptyset),$$

we are totally back to the track of the proof for the stationary case (Theorem 1.3, proved in Samorodnitsky and Shen (2013b)). Here we list the corresponding forms that the lemmas should take under the stationarity of increments.

Lemma 2.7. *Let \mathbf{X} be a stochastic process. If condition (3) in Theorem 2.5 is satisfied, then for any h, d, \mathbf{t} and \mathbf{I} defined as before, with probability 1, $A_{\mathbf{t}, \mathbf{I}}^{h,d}(\mathbf{X})$ is either the empty set or an infinite set, in which case $\inf(A_{\mathbf{t}, \mathbf{I}}^{h,d}(\mathbf{X})) = -\infty$ and $\sup(A_{\mathbf{t}, \mathbf{I}}^{h,d}(\mathbf{X})) = \infty$.*

Lemma 2.8. *Given $h \in \mathbb{R}$, if for any $\Delta > 0, d \geq 2\Delta$, \mathbf{t} and \mathbf{I} defined as before, $p_{\mathbf{t}, \mathbf{I}, a, \Delta}^{h, d}(\mathbf{X})$ is always constant on a , then the process \mathbf{X} is of stationary increments.*

Lemma 2.9. *Assume that for any doubly intrinsic location functional $L \in \mathcal{D}$, any interval $I \in \mathcal{I}$, $L(\mathbf{X}, I)$ admits a density function $f_{\mathbf{X}, I}(t)$ in \dot{I} , which satisfies the total variation constraints on I . Then $p_{\mathbf{t}, \mathbf{I}, a, \Delta}^{h, d}(\mathbf{X})$ is constant on a for any $\Delta > 0, d \geq 2\Delta$, \mathbf{t} and \mathbf{I} defined as before.*

Lemma 2.7 gives us the right to decompose the path space and focus on only one given h . Lemma 2.9 and Lemma 2.8 then lead to the desired result in a straightforward way.

3. DEFINITION OF LOCAL INTRINSIC LOCATION FUNCTIONAL AND REPRESENTATION BY ORDERED SET

The results reviewed in Section 1 showed how closely the concept of intrinsic location functional is related to stationarity. In some sense, the total variation constraints for intrinsic location functionals are just stationarity itself viewed from a different perspective. However, if one only considers the total variation constraint for intervals with a particular length, condition (4) and (5) in Definition 1.1 may appear unnecessarily restrictive: in order to get the total variation constraint for the intervals with this length, one needs to introduce the relationships between intervals with all different lengths. Therefore it is interesting to check if we can adjust the definition of intrinsic location functional, so that it can be defined only for intervals with the given length, while assuring that the total variation constraints still hold for the intervals with this length. It turns out that a reasonable way for this purpose is to define the following object, which we name as “local intrinsic location functional”.

Definition 3.1. *Let $T > 0$ be given. A mapping $L_T : H \times \mathbb{R} \rightarrow \mathbb{R} \cup \{\infty\}$ is called a local intrinsic location functional with related length T , if it satisfies the following conditions.*

- (1) *For every $a \in \mathbb{R}$, the map $L_T(\cdot, a) : H \rightarrow \mathbb{R} \cup \{\infty\}$ is measurable.*

(2) For every $f \in H$ and $a \in \mathbb{R}$, $L_T(f, a) \in [a, a + T] \cup \{\infty\}$.

(3) For every $f \in H$, $a \in \mathbb{R}$ and $c \in \mathbb{R}$,

$$L_T(f, a) = L_T(\theta_c f, a - c) + c,$$

where $\infty + c = \infty$.

(4) For every $f \in H$ and $a, b \in \mathbb{R}$, $L_T(f, a) \in [b, b + T]$ implies that either $L_T(f, b) = L_T(f, a)$, or $L_T(f, b) \in [b, b + T] \setminus [a, a + T]$.

The first three conditions are the same as in the definition of intrinsic location functional. The condition (4) is new and replaces both condition (4) and (5) in Definition 1.1. Intuitively, it first requires that if the locations for two intervals with the same length both fall into the intersection of these two intervals, then they must agree. This is a counterpart of condition (4) (stability under restriction) in Definition 1.1, but now only explicitly involving intervals with one fixed length. The second possibility in condition (4) says that if the location for the first interval is located in the second interval yet is no longer the corresponding location for the second interval, then it must be replaced by another point which is located in the second interval but outside the first interval. In particular, the corresponding location for the second interval can not take value ∞ . In this sense, the second part of condition (4) actually serves as an alternative of condition (5) (consistency of existence) in Definition 1.1.

It is not difficult to see that if we restrict the definition of an intrinsic location functional to intervals with a fixed length, then it automatically gives out a local intrinsic location functional:

Example 3.2. Let $L : H \times \mathcal{I} \rightarrow \mathbb{R} \cup \{\infty\}$ be an intrinsic location functional. Then it is easy to check that for any fixed length $T > 0$, L_T defined by

$$L_T(f, a) = L(f, [a, a + T]),$$

$f \in H, a \in \mathbb{R}$ is a local intrinsic location functional.

On the other hand, a natural “extension” of a local intrinsic location functional to intervals with different lengths does not necessarily give out an intrinsic location functional, as shown by the following example.

Example 3.3. Let $H = \mathcal{C}(\mathbb{R})$, $l > 0$, $L_T(f, a)$ be the first hitting time to a fixed level h in the interval $[a, a + T]$, provided that its distance to the left end point of the interval is at most l . That is,

$$L_T(f, a) = \inf\{t \in [a, a + T] : f(t) = h, t \leq a + l\}.$$

Then L_T is a local intrinsic location functional. However, its “natural” extension, $L(f, [a, b]) := \inf\{t \in [a, b] : f(t) = h, t \leq a + l\}$ is not an intrinsic location functional. To see this, notice that the existence of such a location in an interval with length T does not guarantee its existence for all the larger intervals containing it, since the location may fail to remain close enough to the left end point when the interval expands.

It turns out that despite the large variety covered by the concept of local intrinsic location functional, they all correspond to the idea of taking the maximal element in a random set, ordered according to some specific rule.

Theorem 3.4. *Let H be defined as before. A mapping $L_T = L_T(f, a)$ from $H \times \mathbb{R}$ to $\mathbb{R} \cup \{\infty\}$ is a local intrinsic location functional with related length T , if and only if*

- (1) $L_T(\cdot, a)$ is measurable for $a \in \mathbb{R}$;
- (2) For each function $f \in H$, there exists a subset of \mathbb{R} denoted as $S(f)$ and a partial order \preceq on it, satisfying:
 - (a) For any $c \in \mathbb{R}$, $S(f) = S(\theta_c f) + c$;
 - (b) For any $c \in \mathbb{R}$ and any $t_1, t_2 \in S(f)$, $t_1 \preceq t_2$ implies $t_1 - c \preceq t_2 - c$ in $S(\theta_c f)$,
such that for any $a \in \mathbb{R}$, either $S(f) \cap [a, a + T] = \emptyset$, in which case $L_T(f, a) = \infty$, or $L_T(f, a)$ is the maximal element in $S(f) \cap [a, a + T]$ according to \preceq .

Proof. It is easy to check that the measurability of $L_T(\cdot, a)$ for $a \in \mathbb{R}$ and the existence of such an ordered set $S(f)$ for $f \in H$ guarantee that L_T is a local intrinsic location functional. For the other direction, let L_T be a local intrinsic location functional with related length T . For each path f , define a set

$$S(f) = \{t \in \mathbb{R} : t = L_T(f, a) \text{ for some } a \in \mathbb{R}\}.$$

Thus $S(f)$ is the set of all the points which is chosen as the location for some interval with length T . From now on we fix the function f and simplify the notation $S(f)$ as S . We introduce the following partial binary relation on S . For two points $x, y \in S$, say $x \preceq_0 y$ if and only if there exists an interval $I_{x,y} = [a_{x,y}, a_{x,y} + T]$, such that $x, y \in I_{x,y}$ and $L_T(f, a_{x,y}) = y$. In another word, $x \preceq_0 y$ if and only if some interval with length T containing both of them “chooses” y rather than x to be its corresponding location. Then we complete \preceq_0 by taking the smallest transitive binary relation containing it, denoted as \preceq . We claim that such defined \preceq is actually a partial order on the set S .

The reflexivity is clear: by definition, $x \preceq x, \forall x \in S$. The transitivity is also guaranteed by construction. Therefore the only thing left is to check the antisymmetry: if $x \preceq y$ and $y \preceq x$, then $x = y$. To this end, firstly notice that the construction of the binary relation \preceq_0 guarantees that it is always antisymmetric before being extended to \preceq . That is, $x \preceq_0 y$ and $y \preceq_0 x$ implies $x = y$. Now assume $x \neq y$, $x \preceq y$ and $y \preceq x$, then there is a loop: $x = t_0 \preceq_0 t_1 \preceq_0 \dots \preceq_0 y = t_n \preceq_0 t_{n+1} \preceq_0 \dots \preceq_0 t_{n+m-1} \preceq_0 t_{n+m} = x$ for some positive integers m, n , and points $t_0, t_1, \dots, t_{n+m-1}, t_{n+m} = t_0$ satisfying $|t_{i+1} - t_i| \leq T$ for any $i = 0, \dots, n + m - 1$.

To deal with this loop, notice that we have the proposition below, which states that if two points within a distance no larger than T have a relation \preceq between them, then there must be a direct relation given by \preceq_0 . They can not be only related through a chain of “ \preceq ” via other points.

Lemma 3.5. *Let the relations \preceq_0 and \preceq be as defined above. Then $t_1 \preceq t_2$ and $|t_2 - t_1| \leq T$ imply $t_1 \preceq_0 t_2$ or $t_2 \preceq_0 t_1$.*

Proof. Proof by contradiction. Without loss of generality, assume there are two points $t_1, t_2 \in S$, $t_1 < t_2$, $t_2 - t_1 \leq T$, there exist points $s_0, s_1, \dots, s_n, s_{n+1}$ such that $s_0 = t_1 \preceq_0 s_1 \preceq_0 \dots \preceq_0 s_n \preceq_0 s_{n+1} = t_2$, however, there is no direct relation given by \preceq_0 between t_1 and t_2 . That is, every interval with length T containing the interval $[t_1, t_2]$ have neither t_1 nor t_2 as its corresponding location. Since $t_1 \in S$, there is $a \in \mathbb{R}$, such that $L_T(f, a) = t_1$. The interval $[a, a + T]$ can not include t_2 , otherwise $t_2 \preceq_0 t_1$. Therefore $a + T < t_2$.

Consider $L_T(f, t_2 - T)$. Because $L_T(f, a) = t_1 \in [a, a + T] \cap [t_2 - T, t_2]$, the condition (4) in the definition of local intrinsic location functional rules out the possibility that $L_T(f, t_2 - T) = \infty$ or $L_T(f, t_2 - T) \in [a, a + T] \cap [t_2 - T, t_2]$. Thus $L_T(f, t_2 - T) \in (a + T, t_2] \subseteq (t_1, t_2]$. It can not be t_2 either since then $t_1 \preceq_0 t_2$. As a result, $L_T(f, t_2 - T) \in (t_1, t_2)$. Denote $L_T(f, t_2 - T)$ by t_3 . Then $t_3 \in S$ and by definition $t_1 \preceq_0 t_3$ and $t_2 \preceq_0 t_3$.

Consider the intervals $[s_j, s_{j+1})$ for $j = 0, \dots, n$ which satisfies $s_j < s_{j+1}$. Clearly, their union covers the interval $[t_1, t_2)$, therefore also the point t_3 . Assume $t_3 \in [s_k, s_{k+1})$. There are two cases. Case 1: $s_{k+1} \leq t_2$. Since $s_k \preceq_0 s_{k+1}$, there is a real number a_1 , such that $s_k \in [a_1, a_1 + T]$ and $L_T(f, a_1) = s_{k+1}$. Similarly, since $t_2 \preceq_0 t_3$, there is a real number a_2 such that $t_2 \in [a_2, a_2 + T]$ and $L_T(f, a_2) = t_3$. However $s_{k+1} \leq t_2$ implies that both s_{k+1} and t_3 are in the interval $[a_1, a_1 + T] \cap [a_2, a_2 + T]$, thus contradicting with the definition of local intrinsic location functional. Case 2: $s_{k+1} > t_2$. In this case, notice that $t_2 \in S$, so there exists a_3 such that $L_T(f, a_3) = t_2$. However, since \preceq_0 is antisymmetric, $t_2 \preceq_0 t_3$ implies that $t_3 \not\preceq_0 t_2$, so $a_3 > t_3$. Now both t_2 and s_{k+1} are in the interval $[a_1, a_1 + T] \cap [a_3, a_3 + T]$, yet L_T gives out different locations, contradiction again. To conclude, the assumption at the beginning of the proof can not hold, and the lemma is proved. \square

Now we turn back to the loop and prove the following result: there exist $i_1, i_2, i_3 \in \{0, \dots, n + m - 1\}$, such that $t_{i_1} \preceq_0 t_{i_2} \preceq_0 t_{i_3} \preceq_0 t_{i_1}$. Consider the rightmost point in the set $\{t_i\}_{i=0, \dots, n+m-1}$, denoted as $t_j := \max_{i=0}^{n+m-1} t_i$. Notice that $t_{j-1} < t_j$, $t_{j+1} < t_j$, therefore $|t_{j+1} - t_{j-1}| < T$, and $t_{j-1} \preceq_0 t_j \preceq_0 t_{j+1}$ (define $t_{-1} = t_{n+m-1}$). By lemma 3.5 there is a relation \preceq_0 between t_{j+1} and t_{j-1} . If $t_{j+1} \preceq_0 t_{j-1}$, we already have a loop with three terms as desired. If $t_{j-1} \preceq_0 t_{j+1}$, then consider the set $\{t_i\}_{i=0, \dots, n+m-1, i \neq j}$. It is also a loop as the set $\{t_i\}_{i=0, \dots, n+m-1}$ by which we started, now with one less term. An iteration of this procedure finally decreases the size of the set to 3, so we find a loop with 3 terms again.

The existence of a loop with 3 terms, however, contradicts with the definition of the relation \preceq_0 . To see this, without loss of generality, suppose that we have $t_1 < t_2 < t_3$ satisfying $t_1 \preceq_0 t_2 \preceq_0 t_3 \preceq_0 t_1$. This means that there

exists $a, b \in \mathbb{R}$, such that $t_1, t_3 \in [a, a+T]$ and $L_T(f, a) = t_1$, $t_1, t_2 \in [b, b+T]$ and $L_T(f, b) = t_2$. However, the fact that $t_1, t_2 \in [a, a+T] \cap [b, b+T]$, yet $L_T(f, a)$ and $L_T(f, b)$ are not equal contradicts with the definition of local intrinsic location functional.

In total, we have seen that a loop of relation \preceq_0 , therefore also \preceq , is not possible. Thus the antisymmetry is proved. The relation \preceq is a partial order. Finally, it is clear by the construction of the partially ordered set $(S(f), \preceq)$ that either $S(f) \cap [a, a+T] = \emptyset$, in which case $L_T(f, a) = \infty$, or $L_T(f, a) \in S(f) \cap [a, a+T]$, in which case $s \preceq L_T(f, a)$ for all $s \in S(f) \cap [a, a+T]$. \square

Remark 3.6. The partial order in the theorem has the special property that there exists a unique maximal element over any interval with length T . In this sense it behaves like a total order. Indeed, by order extension principle, the partial order \preceq can always be extended to a total order on $S(f)$, and it is clear that we can do it in a shift-invariant way, so that the resulting total order also satisfies the conditions in Theorem 3.4. Nonetheless, here we would like to keep \preceq a partial order for generality.

A similar reasoning allows us to derive the ordered set representation for intrinsic location functionals.

Corollary 3.7. *Let H, \mathcal{I} be defined as before. A mapping $L = L(f, I)$ from $H \times \mathcal{I}$ to $\mathbb{R} \cup \{\infty\}$ is an intrinsic location functional if and only if*

- (1) $L(\cdot, I)$ is measurable for $I \in \mathcal{I}$;
- (2) For each function $f \in H$, there exists a partially ordered subset of \mathbb{R} , denoted as $(S(f), \preceq_1)$, satisfying:
 - (a) For any $c \in \mathbb{R}$, $S(f) = S(\theta_c f) + c$;
 - (b) For any $c \in \mathbb{R}$ and any $t_1, t_2 \in S(f)$, $t_1 \preceq_1 t_2$ implies $t_1 - c \preceq_1 t_2 - c$ in $S(\theta_c f)$,
such that for any $I \in \mathcal{I}$, either $S(f) \cap I = \emptyset$, in which case $L(f, I) = \infty$, or $L(f, I)$ is the maximal element in $S(f) \cap I$ according to \preceq_1 .

Proof. Again, it is routine to check the “if” direction. For the other direction, define $S(f) := \{t \in \mathbb{R} : L(f, I) = t \text{ for some } I \in \mathcal{I}\}$ and the binary relation \preceq_1 on $S(f)$: $x \preceq_1 y$ if and only if there exists an interval $I \in \mathcal{I}$ such that

$x, y \in I$ and $L(f, I) = y$. The argument goes through in the same way, and is actually simpler, since such defined \preceq_1 is now directly a partial order. \square

Example 3.8. Let H be the space of all upper semi-continuous functions on \mathbb{R} . The location of the path supremum $\tau_{f,I} := \inf\{t \in I : f(t) = \sup_{s \in I} f(s)\}$, $f \in H, I \in \mathcal{I}$ is an intrinsic location functional. It corresponds to an ordered set $(S(f), \preceq)$, where $S(f) = S^1(f) \cup S^2(f)$, $S^1(f)$ is the union of the set of local maxima of f , and $S^2(f) := \{t \in \mathbb{R} : t = \sup_{s \in [t-T, t]}(f(s)) \text{ or } t = \sup_{s \in [t, t+T]}(f(s))\}$. “ \preceq ” is firstly ordered by the value of the function at the points and in case of a tie, inversely ordered by the location (that is, the locations on the left receive high orders).

Example 3.9. Let H be the space of all continuous functions on \mathbb{R} . The first hitting time of a level l over an interval I , defined by $T_{f,I}^l := \inf\{t \in I : f(t) = l\}$ is also an intrinsic location functional. The ordered set $(S(f), \preceq)$ is now given by $S(f) = f^{-1}(l)$ and the inverse order on the real line.

It is clear that the partially ordered random set representation of a local intrinsic location functional or an intrinsic location functional can not be unique, since one can always add irrelevant points to $S(f)$ and assign them very low orders, so that the added points are actually never chosen as the location for any interval. However, there exists a unique minimal representation, as indicated by the proof of Theorem 3.4.

Corollary 3.10. *Let L be a local intrinsic location functional (resp. intrinsic location functional) with path space H . There exists a partially ordered set $(S(f), \preceq)$ for each function $f \in H$, satisfying the conditions in Theorem 3.4 (resp. Corollary 3.7), such that for any other partially ordered set $(S'(f), \preceq')$ also satisfying the same conditions,*

$$S(f) \subseteq S'(f)$$

and

$$s_1, s_2 \in S(f), s_1 \preceq s_2 \text{ implies } s_1 \preceq' s_2 \text{ in } S'(f).$$

The proof is easy and omitted. Notice that we do not only know the existence of the minimal representation, it is actually straightforward to

write it down explicitly. For a local intrinsic location functional L_T with related length $T > 0$ and $f \in H$, $S(f) = \{t : L_T(f, a) = t \text{ for some } a \in \mathbb{R}\}$, and \preceq is the smallest partial order such that $s_1 \preceq s_2$ for all s_1, s_2 satisfying $s_1 \in [a, a + T]$ and $L_T(f, a) = s_2$ for some $a \in \mathbb{R}$. Similarly, for an intrinsic location functional L and $f \in H$, $S(f) = \{t : L(f, I) = t \text{ for some } I \in \mathcal{I}\}$, and \preceq is given by $s_1 \preceq s_2$ if $s_1 \in I$ and $L(f, I) = s_2$ for some $I \in \mathcal{I}$.

4. EXTENSION AND RESTRICTION

The ordered set representation provides powerful tools for us to clarify the link between intrinsic location functional and local intrinsic location functional. The theorem below shows that a local intrinsic location functional is “almost” just a “local” version of an intrinsic location functional.

We call a mapping L from $H \times \mathcal{I}$ to $\mathbb{R} \cup \{\infty\}$ a “pre-intrinsic location functional”, if it satisfies all the defining properties in Definition 1.1 except for the measurability condition (1). In another word, a pre-intrinsic location functional becomes an intrinsic location functional once it is measurable for all compact intervals $I \in \mathcal{I}$.

Theorem 4.1. *Let L be an intrinsic location functional. Then for any $T > 0$,*

$$(4.1) \quad L_T(f, a) := L(f, [a, a + T])$$

is a local intrinsic location functional. Conversely, let L_T be a local intrinsic location functional. Then there exists a pre-intrinsic location functional L , such that (4.1) holds for all $f \in H$ and $a \in \mathbb{R}$.

Proof. The fact that a restricted intrinsic location functional is a local intrinsic location functional can be easily checked either by their definitions or by the ordered set representation. For the other direction, suppose we have a local intrinsic location functional L_T , with the partially ordered set $(S(f), \preceq)$ for each $f \in H$. By the order extension principle, $(S(f), \preceq)$ can always be extended, in a shift-invariant way, to a totally ordered set $(S(f), \preceq_1)$, which is, of course, a special partially ordered set. Define $L(f, I)$ for any $I \in \mathcal{I}$ by taking the maximal element in I of $S(f)$ according to \preceq_1 : $L(f, I) \in S(f)$

and $s \preceq_1 L(f, I)$ for all $s \in S(f) \cap I$, then by Corollary 3.7 such defined L is a pre-intrinsic location functional. \square

Notice, however, that we have not touched the measurability issue and claimed that each local intrinsic location functional necessarily has an intrinsic location functional extension. The problem of measurability is highly nontrivial and in general, the measurability of a local intrinsic location functional for intervals with a single fixed length may not be enough to guarantee the measurability of its extensions with all different interval lengths. Instead, we prove the following result, which shows that there always exists an intrinsic location functional which agrees almost surely with the given local intrinsic location functional for any stationary process in the interior of any interval with the fixed length.

Proposition 4.2. *Let $L_T : H \times \mathbb{R} \rightarrow \mathbb{R} \cup \{\infty\}$ be a local intrinsic location functional with related length T . Then there exists an intrinsic location functional $L : H \times \mathcal{I} \rightarrow \mathbb{R} \cup \{\infty\}$, such that for any $a \in \mathbb{R}$ and stationary process \mathbf{X} with paths in H ,*

$$\mathbb{P}[L_T(\mathbf{X}, a) \neq L(\mathbf{X}, [a, a + T])],$$

$$L_T(\mathbf{X}, a) \in (a, a + T) \text{ or } L(\mathbf{X}, [a, a + T]) \in (a, a + T) = 0.$$

Before we go to the proof of Proposition 4.2, let us first look at a useful lemma.

Lemma 4.3. *Let L_T be a local intrinsic location functional defined on $H \times \mathbb{R}$. Then*

- (1) *For any $f \in H$, any $a < b$ such that $L_T(f, a) \neq \infty$ and $L_T(f, b) \neq \infty$, $L_T(f, a) \leq L_T(f, b)$.*
- (2) *If $L_T(f, a) = L_T(f, b) = t \neq \infty$, then $L_T(f, c) = t$ for any $c \in [a, b]$.*
- (3) *If $a < b$, $b - a \leq T$ and $L_T(f, a) = L_T(f, b) = \infty$, then $L_T(f, c) = \infty$ for all $c \in [a, b]$.*

Proof. Suppose for some $a < b$, $L_T(f, b) < L_T(f, a) < \infty$. Then both $L_T(f, a)$ and $L_T(f, b)$ are in the interval $[a, a + T] \cap [b, b + T]$. However, by the definition of local intrinsic location functional, this implies that they

must be equal. Thus the first claim of the proposition is proved. Now assume $L_T(f, a) = L_T(f, b) = t \neq \infty$. Then $t \in [a, a+T] \cap [b, b+T] = [b, a+T] \neq \phi$. For any $c \in [a, b]$, $[c, c+T] \supseteq [b, a+T]$, hence $t \in [a, a+T] \cap [c, c+T]$. By definition of local intrinsic location functional, $L_T(f, c) \neq \infty$. Then by the first claim of the proposition, $t = L_T(f, a) \leq L_T(f, c) \leq L_T(f, b) = t$. Therefore $L_T(f, c) = t$ as well. Finally, if $a < b$, $b - a \leq T$, then for any $c \in [a, b]$, $[c, c+T] \subset [a, a+T] \cup [b, b+T]$. If $L_T(f, a) = L_T(f, b) = \infty$, then by Theorem 3.4, $[a, a+T] \cap S(f) = [b, b+T] \cap S(f) = \phi$, where $S(f)$ is a set of points corresponding to L_T . As a result, $[c, c+T] \cap S(f) = \phi$, which, going back to L_T , means that $L_T(f, c) = \infty$.

□

Proof of Proposition 4.2. For any function $f \in H$, define the sets

$$S_1(f) := \{t \in \mathbb{R} : \exists(x, y) \subset \mathbb{R}, \text{ s.t. } L_T(f, a) = t, \forall a \in (x, y)\},$$

$$S_2(f) := \{t \in \mathbb{R} \setminus S_1(f) : L_T(f, t) = t \text{ or } L_T(f, t - T) = t\}$$

and $S'(f) = S_1(f) \cup S_2(f)$.

On $S'(f)$ assign a binary relation \preceq_0 : $t_1 \preceq_0 t_2$ if and only if $|t_2 - t_1| < T$ and there exists a real number a satisfying $t_1, t_2 \in [a, a+T]$ such that $L_T(f, a) = t_2$. Notice that the set $S'(f)$ is a subset of the set we constructed in the proof of Theorem 4.1, and \preceq_0 is also a restriction of the corresponding binary relation that we saw before. As a result, one can again extend \preceq_0 to a smallest partial order, still denoted by \preceq .

For function $f \in H$ and a compact interval I , define $L(f, I)$ to be the first element in $S'(f)$ which is maximal in I :

$$L(f, I) = \inf\{t \in S'(f) \cap I : t' \in S'(f) \cap I \text{ and } t \preceq t' \text{ implies } t' = t\}.$$

We can denote the set on the right hand side of the definition above, namely, the set of all the maximal in I points in $S'(f)$, by $M_{f,I}$. Then $L(f, I)$ is simply $\inf(M_{f,I})$, with the tradition that $\inf(\phi) = \infty$. Indeed, this way of choosing the first maximal element is equivalent to assigning an additional order among the maximal elements according to their location, with the left receiving the higher order and the right lower. The resulting new order will then satisfy all the conditions listed in Corollary 3.7, which assures that such

defined $L(f, I)$ is a pre-intrinsic location functional. Thus all that is left is to check the measurability.

Fix $I = [a, b]$ with $|I| = b - a > T$ and $f \in H$. The event $\{L(f, I) \leq s\}$ is $\{a \in M_{f,I}\}$ if $s = a$, $\{a \in M_{f,I}\} \cup \{M_{f,I} \cap (a, s] \neq \emptyset\}$ if $s \in (a, b)$, and $\{a \in M_{f,I}\} \cup \{M_{f,I} \cap (a, s] \neq \emptyset\} \cup \{b \in M_{f,I}\}$ if $s = b$. Therefore it suffices to verify the measurability for each of these sets.

Lemma 4.4. *Let $I = [a, b]$, $b - a > T$ and $t \in (a, b)$, then $t \in M_{f,I}$ if and only if for some sequences $\{t_{1n}\}_{n=1,2,\dots}$ and $\{t_{2n}\}_{n=1,2,\dots}$ such that $t_{1n} \rightarrow t$ and $t_{2n} \rightarrow t$ as $n \rightarrow \infty$, $L_T(f, a \vee (t_{1n} - T)) = L_T(f, (b - T) \wedge t_{2n}) = t$ holds for $n = 1, 2, \dots$.*

Proof. Firstly assume that $t \in M_{f,I} \cap (a, b)$. If $a \leq t - T$, then for any $s \in (t, (t + T) \wedge b)$, $[s - T, s] \subset (a, b)$, and $t \in (s - T, s)$. By the maximality of t under the partial order \preceq , $L_T(f, s - T) = t$. Therefore we only need to take $\{t_{1n}\}_{n=1,2,\dots}$ a decreasing sequence converging to t with $t_{11} < (t + T) \wedge b$ to have $L_T(f, t_{1n} - T) = t$. If $a > t - T$, then the maximality implies that $L_T(f, a) = t$. Combining these two cases, there always exist $\{t_{1n}\}_{n=1,2,\dots}$ such that $L_T(f, a \vee (t_{1n} - T)) = t$. Symmetrically we have $L_T(f, (b - T) \wedge t_{2n}) = t$ for some $\{t_{2n}\}_{n=1,2,\dots}$.

The case where $t \notin S'(f)$ being trivial, now suppose $t \in S'(f) \cap (a, b)$ but $t \notin M_{f,I}$. Then there exists $s \in (t - T, t + T) \cap [a, b]$ such that $t \preceq_0 s$. Without loss of generality, assume that $s < t$. Then for any $r \in [t - T, s]$, $L_T(f, r) \neq t$, since otherwise $s \preceq_0 t$. Therefore there does not exist a sequence $\{t_{1n}\}_{n=1,2,\dots}$ such that $L_T(f, a \vee (t_{1n} - T)) = t$. The lemma is proved. \square

For any x, y such that $a \leq x < y \leq b$, denote by $E_I(x, y)$ the event $L_T(f, a \vee (y - T)) = L_T(f, (b - T) \wedge x) \neq \infty$. For $r, s \in (a, b)$ and $m = 1, 2, \dots$, define event $E_{I,m}(r, s) = \bigcup_{i=1}^{2^m-1} E_I(r + \frac{(i-1)(s-r)}{2^m}, r + \frac{(i+1)(s-r)}{2^m})$. Consider the set

$$\begin{aligned} E(I, r, s) &:= \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} E_{I,m}(r, s) \\ &= \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} \bigcup_{i=1}^{2^m-1} E_I(r + \frac{(i-1)(s-r)}{2^m}, r + \frac{(i+1)(s-r)}{2^m}). \end{aligned}$$

It is clearly measurable. Suppose there is a point $t \in (r, s)$ in $M_{f,I}$. For any m large enough, let i' be an index satisfying $t \in (r + \frac{(i'-1)(s-r)}{2^m}, r + \frac{(i'+1)(s-r)}{2^m})$.

Then event $E_I(r + \frac{(i'-1)(s-r)}{2^m}, r + \frac{(i'+1)(s-r)}{2^m})$ holds. Consequently $E_{I,m}(r, s)$ holds hence $E(I, r, s)$ also holds. Thus $\{M_{f,I} \cap (r, s) \neq \phi\} \subseteq E(I, r, s)$. On the other hand, suppose $E(I, r, s)$ is realized. Then for all m large enough, $E_I(r + \frac{(i-1)(s-r)}{2^m}, r + \frac{(i+1)(s-r)}{2^m})$ holds for some $i = 1, \dots, 2^m - 1$. Denote by J_m the set of indices $i = 1, \dots, 2^m - 1$ for which $E_I(r + \frac{(i-1)(s-r)}{2^m}, r + \frac{(i+1)(s-r)}{2^m})$ holds, and $B_m := \bigcup_{i \in J_m} [r + \frac{(i-1)(s-r)}{2^m}, r + \frac{(i+1)(s-r)}{2^m}]$. It is easy to check by definition that B_m is a decreasing sequence of closed sets, thus there exists some point $t \in [r, s]$ which is covered by infinite members in $\{B_m\}_{m=0,1,\dots}$, therefore also infinite number of intervals forming $B_m, m = 0, 1, \dots$. Let $\{I_{m_j} = [a_{m_j}, b_{m_j}]\}_{j=1,2,\dots}$ be such a sequence always covering t . Notice that $a_{m_j} \rightarrow t$ and $b_{m_j} \rightarrow t$ as $j \rightarrow \infty$. Moreover, $E_I(a_{m_j}, b_{m_j})$ holds for all $j = 1, 2, \dots$ by construction. Thus by Lemma 4.4 $t \in M_{f,I}$. Thus we have

$$\{M_{f,I} \cap (r, s) \neq \phi\} \subseteq E(I, r, s) \subseteq \{M_{f,I} \cap [r, s] \neq \phi\},$$

which implies

$$E(I, r, s) \cup \{r \in M_{f,I}\} \cup \{s \in M_{f,I}\} = \{M_{f,I} \cap [r, s] \neq \phi\}.$$

It is easy to check that $\{r \in M_{f,I}\}$ and $\{s \in M_{f,I}\}$ are measurable. $\{r \in M_{f,I}\}$, for example, can only happen if $r \in M_{f,I} \cap S_1(f)$, which is then equivalent to $\bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} E_I(r - \frac{1}{m}, r + \frac{1}{m})$. As a result, $\{M_{f,I} \cap [r, s] \neq \phi\}$ is measurable for any r, s in the interior of $[a, b]$. It is then trivial to see the measurability of $\{M_{f,I} \cap (a, s] \neq \phi\}$ for $s \in (a, b)$ or $\{M_{f,I} \cap (a, s) \neq \phi\}$ for $s = b$ by taking a countable union. The case for the two endpoints a and b can be checked directly. The measurability of $a \in M_{f,I}$, for instance, is verified once we observe that $a \in M_{f,I} \cap S_1(f)$ if and only if there exists a sequence $\{s_n\}_{n=1,2,\dots}$ such that $s_n \uparrow a$ and $L_T(f, s_n) = a$ for $n = 1, 2, \dots$. $a \in M_{f,I} \cap S_2(f)$, of course, if and only if $L_T(f, a) = a$.

For the case of $I = [a, b]$ with $|I| = T$, the key is to notice that $L(f, I) = t \in (a, b)$ if and only if there exists a positive integer n such that $L(f, I_{1n}) = t$ or $L(f, I_{2n}) = t$, where $I_{1n} = [a - \frac{1}{n}, b]$, $I_{2n} = [a, b + \frac{1}{n}]$, and $L(f, I_{1n})$ and $L(f, I_{2n})$ are defined as above for $|I| > T$. Thus for any $s \in (a, b)$,

$$\{L(f, I) \in [a, s]\} = \{a \in M_{f,I}\} \cup \left(\bigcup_{n=1}^{\infty} \{L(f, I_{in}) \in (a, s], i = 1 \text{ or } 2\} \right)$$

is measurable. The cases with $s = a$ or $s = b$ are not much different from before.

Finally if $I = [a, b]$ with $|I| < T$, $L(f, I) = t \in (a, b)$ is equivalent to the existence of three points $x, y \in \mathbb{Q}$ and $z \in (\mathbb{Q} \cap [b - T, a]) \cup \{a, b - T\}$, such that $L_T(f, x) = L_T(f, y) = L_T(f, z) = t$. It is not difficult to check this equivalence. Intuitively, the existence of x and y assures that $t \in S'(f)$, while the existence of z guarantees the maximality of t in I . The countability of the rational set then leads to the measurability. We skip the details.

Combining the three cases proves the measurability of $L(\cdot, I)$ for any compact interval I , as desired. L is thus an intrinsic location functional. The last thing in the proof is therefore to show the relationship between $L_T(\mathbf{X}, a)$ and $L(\mathbf{X}, [a, a + T])$ claimed in the proposition.

Let \mathbf{X} be any stationary process with paths in H . Firstly, assume $L(\mathbf{X}, [a, a + T]) = t \in (a, a + T)$ but $L_T(\mathbf{X}, a) \neq t$. Then $L_T(\mathbf{X}, a) \notin S'(\mathbf{X})$, since otherwise t and $L_T(\mathbf{X}, a)$ are both in $S'(\mathbf{X})$, $|t - L_T(\mathbf{X}, a)| < T$ and by definition of \preceq_0 , $t \preceq_0 L_T(\mathbf{X}, a)$, contradicting the maximality of $L(\mathbf{X}, [a, a + T])$. By the same reasoning, if $L_T(\mathbf{X}, a) \in (a, a + T)$ but $L(\mathbf{X}, [a, a + T]) \neq L_T(\mathbf{X}, a)$ then $L_T(\mathbf{X}, a) \notin S'(\mathbf{X})$. Together, we have

$$\begin{aligned} & \{L_T(\mathbf{X}, a) \neq L(\mathbf{X}, [a, a + T])\}, \\ & L_T(\mathbf{X}, a) \in (a, a + T) \text{ or } L(\mathbf{X}, [a, a + T]) \in (a, a + T) \} \\ & \subseteq \{L_T(\mathbf{X}, a) \notin S'(\mathbf{X})\}. \end{aligned}$$

Notice that if $L_T(\mathbf{X}, a) = a$ or $L_T(\mathbf{X}, a) = a + T$, then $L_T(\mathbf{X}, a) \in S'(\mathbf{X})$ automatically. By the definition of $S'(X)$, $L_T(\mathbf{X}, a) \notin S'(\mathbf{X})$ if and only if $L_T(\mathbf{X}, a) \in (a, a + T)$ and $L_T(\mathbf{X}, a) \neq L_T(\mathbf{X}, b)$ for any $b \neq a$, which is equivalent to $L_T(\mathbf{X}, a) \neq L_T(\mathbf{X}, b)$ for any $b \neq a$, $b \in \mathbb{Q}$ by Lemma 4.3. Thus $\{L_T(\mathbf{X}, a) \notin S'(\mathbf{X})\}$ is measurable. Now we show that $\mathbb{P}(L_T(\mathbf{X}, a) \notin S'(\mathbf{X})) = 0$. Assume $\mathbb{P}(L_T(\mathbf{X}, a) \in (a, a + T) \setminus S'(\mathbf{X})) > 0$. Then there exists $\Delta > 0$, such that $\mathbb{P}(L_T(\mathbf{X}, a) \in (a + \Delta, a + T - \Delta) \setminus S'(\mathbf{X})) =: \delta > 0$. Take $\epsilon < \Delta/(\lfloor 1/\delta \rfloor)$ and compact intervals $I_i = [a + i\epsilon, a + i\epsilon + T]$ for $i = 0, 1, \dots, \lfloor 1/\delta \rfloor$, where " $\lfloor \cdot \rfloor$ " refers to the largest integer smaller or equal to the argument. By construction, for any $i, j = 0, 1, \dots, \lfloor 1/\delta \rfloor$, $I_i \cap I_j \supset [a + i\epsilon + \Delta, a + i\epsilon + T - \Delta] \cup [a + j\epsilon + \Delta, a + j\epsilon + T - \Delta]$. This, however, implies

that the events $E_i := \{L_T(\mathbf{X}, a + i\epsilon) \in (a + i\epsilon + \Delta, a + i\epsilon + T - \Delta) \setminus S'(\mathbf{X})\}$ must be disjoint for different i . Otherwise, suppose E_i and E_j holds for some $i < j$. Then since both $L_T(\mathbf{X}, a + i\epsilon)$ and $L_T(\mathbf{X}, a + j\epsilon)$ are in the intersection of I_i and I_j , they must be equal. Lemma 4.3 then implies that $L_T(\mathbf{X}, a') = L_T(\mathbf{X}, a + i\epsilon)$ for all $a' \in [a + i\epsilon, a + j\epsilon]$. This contradicts with E_i , which requires that $L_T(\mathbf{X}, a + i\epsilon) \notin S'(\mathbf{X})$. By stationarity, $\mathbb{P}(E_i) = \mathbb{P}(L_T(\mathbf{X}, a) \in (a + \Delta, a + T - \Delta) \setminus S(\mathbf{X})) = \delta, i = 0, 1, \dots, \lfloor 1/\delta \rfloor$. Then

$$\mathbb{P}\left(\bigcup_{i=0}^{\lfloor 1/\delta \rfloor} E_i\right) = \delta \cdot (\lfloor 1/\delta \rfloor + 1) > 1,$$

which clearly shows a contradiction. As a result, $\mathbb{P}(L_T(\mathbf{X}, a) \notin S'(\mathbf{X})) = 0$ and the proof of the proposition is complete. \square

The importance of Proposition 4.2 resides in the fact that most of the distributional properties of intrinsic location functionals proved in Samorodnitsky and Shen (2013b) can now be transformed automatically to local intrinsic location functionals. In particular, local intrinsic location functionals always satisfy the total variation constraints. Thus the equivalence between the stationarity, the total variation constraints and the shift invariance of the distributions can be extended to local intrinsic location functionals.

Corollary 4.5. *Let \mathbf{X} be a stochastic process with continuous paths. Let $\mathcal{L}_{loc,T}$ be the set of all local intrinsic location functionals in $\mathcal{C}(\mathbb{R})$ with related length T . Then the followings are equivalent:*

- (1) *The process \mathbf{X} is stationary.*
- (2) *For any $T > 0$, any local intrinsic location functional $L_T \in \mathcal{L}_{loc,T}$, the distribution of $L_T(\mathbf{X}, a) - a$ does not depend on a .*
- (3) *For any $T > 0$, any local intrinsic location functional $L_T \in \mathcal{L}_{loc,T}$ and any $a \in \mathbb{R}$, $L_T(\mathbf{X}, a)$ admits a density function $f_{\mathbf{X},a,T}(t)$ in $(a, a+T)$, which satisfies the total variation constraint on $[a, a+T]$.*

Remark 4.6. A closer examination of the proof of the equivalence theorem in Samorodnitsky and Shen (2013b) shows that the length of the interval does not play any crucial role in the proof of the equivalence between (1) and (2). As a result, (2) in Corollary 4.5 is also equivalent to:

(2') For a fixed $T > 0$, any local intrinsic location functional $L_T \in \mathcal{L}_{loc,T}$, the distribution of $L_T(\mathbf{X}, a) - a$ does not depend on a .

To sum up, while the equivalence between the stationarity and the total variation constraints of the intrinsic location functionals have been established in Samorodnitsky and Shen (2013b), we just extended this result to local intrinsic location functionals, which is more generally defined compared to intrinsic location functionals. Moreover, the local intrinsic location functionals are further identified with the shift-compatible ordered sets of points $(S(\cdot), \preceq)$ on \mathbb{R} as path functionals. Such an identification provides a particularly convenient way to define local intrinsic location functionals.

We complete this section by the following corollary, which examines the relation between intrinsic location functionals and local intrinsic location functionals, from the perspective of the partially ordered sets they correspond to. The proof is easy and omitted.

Corollary 4.7. *Let H, \mathcal{I} be defined as before. Let L be an intrinsic location functional, then $L_T : H \times \mathbb{R} \rightarrow \mathbb{R} \cup \{\infty\}$ defined by $L_T(f, a) = L(f, [a, a+T])$ is a local intrinsic location functional for each $T > 0$. If $(S(\cdot), \preceq_1)$ and $(S_T(\cdot), \preceq_T)$ are the minimal ordered random set representations for L and L_T respectively, then for $f \in H$, $S_T(f) \subseteq S(f)$, and $t_1, t_2 \in S_T(f), t_1 \preceq_T t_2$ implies $t_1 \preceq_1 t_2$ in $S(f)$.*

On the other hand, let $\{L_T\}_{T>0}$ be a family of local intrinsic location functionals, with minimal ordered random set representations $\{(S_T(\cdot), \preceq_T)\}_{T>0}$. Then there exists an intrinsic location functional L such that $L(f, I) = L_{b-a}(f, a)$ for any $I = [a, b] \in \mathcal{I}$ and any $f \in H$, if and only if there exists a partially ordered random set $(S(\cdot), \preceq_1)$ satisfying the same properties as in condition (2) in Corollary 3.7, such that for any $T > 0$, $f \in H$, $S_T(f) \subseteq S(f)$, and $t_1, t_2 \in S_T(f), t_1 \preceq_T t_2$ implies $t_1 \preceq_1 t_2$ in $S(f)$.

5. PATH CHARACTERIZATION

Let L_T be a local intrinsic location functional with related length T . Given any $f \in H$, define $g(x) := L_T(f, x) - x, \forall x \in \mathbb{R}$. Thus $g(x)$ is the distance between L_T and the starting point x of the interval of interest. Notice that

since L_T can take value infinity, $g(x)$ can also be infinity. The following result gives out a characterization of the function g . In another word, it answers the question how we can tell whether a random location is a local intrinsic location functional by looking at the change of its place in the interval as the interval shifts over the real line.

We call a partition satisfying certain property the “roughest”, if all the other partitions satisfying the same property is a refinement of the given partition.

Theorem 5.1. *Let L_T be a local intrinsic location functional discussed before, and g be the function defined above. Then for any $f \in H$, there exists a roughest partition of the real line by intervals (the intervals can be degenerated, and the boundaries of the intervals can be open or closed), such that for any member $I = (a, b), (a, b], [a, b)$ or $[a, b]$ of this partition, exactly one of the followings is true.*

- (1) $b - a \leq T$, and $g(x) = d - x$ for some $d \in [b, a + T]$ and all $x \in I$.
- (2) $g(x) = \infty$ for all $x \in I$.

Moreover, if $g(a) \neq T$ (resp. $g(b) \neq 0$), then $\lim_{x \uparrow a} g(x) = 0$ (resp. $\lim_{x \downarrow b} g(x) = T$). If I is open on a (resp. b), then $g(a) = 0$ (resp. $g(b) = T$).

On the other hand, let L_T be a mapping from $H \times \mathbb{R}$ to $\mathbb{R} \cup \{\infty\}$ such that $L_T(\cdot, a)$ is measurable for any $a \in \mathbb{R}$, and $L_T(f, a) = L_T(\theta_c f, a - c) + c$ for any $a, c \in \mathbb{R}$. If for any function $f \in H$, there always exists a partition P of the real line by intervals satisfying the properties listed above, then L_T is a local intrinsic location functional with related length T .

Roughly speaking, Theorem 5.1 tells us that the function g consists of linear pieces with slope -1 and intervals with value ∞ . The pieces are combined together following the rule that when the interval $[x, x + T]$ shifts along the real line, a location can “disappear” in the interior of the interval only if it is replaced by another location appearing at the right endpoint $x + T$. Symmetrically, a location can only “appear” in the interior of the interval only if it is replacing another location disappearing at the left endpoint x . The actual scenario is a little bit more complicated, since both the replaced and

replacing “location” can be indeed the limit of a sequence of locations, where comes the limits in the formulation of the theorem.

Proof. Let L_T be a local intrinsic location functional with related length T . By Theorem 3.4 and Remark 3.6, for each $f \in H$, there is a set $S(f) \subseteq \mathbb{R}$ and a partial order \preceq on it, satisfying $S(\theta_c f) = S(f) - c$ and $t_1 \preceq t_2$ in $S(f)$ if and only if $t_1 - c \preceq t_2 - c$ in $S(\theta_c f)$ for any $c \in \mathbb{R}$, such that $L_T(f, x)$ is the unique maximal element by \preceq in $S(f) \cap [x, x + T]$ for any $f \in H$ and any $x \in \mathbb{R}$, provided that it exists. For any fixed $x \in \mathbb{R}$, there are two cases. Case 1: $S(f) \cap [x, x + T] = \emptyset$. In this case define $a = \sup\{S(f) \cap (-\infty, x)\}$ and $b = \inf\{S(f) \cap (x + T, \infty)\} - T$. Then a, b are clearly the two boundaries of the largest interval containing x on which $L_T(f, \cdot) = \infty$. Notice that it is possible to have $a = b$, in which case the interval becomes degenerate. Case 2: $S(f) \cap [x, x + T] \neq \emptyset$. In this case define $a = \max\{L_T(f, x) - T, \sup\{y \in \mathbb{R} : y \in S(f), y < L_T(f, x), L_T(f, x) \preceq y\}\}$ and $b = \min\{L_T(f, x), \inf\{y \in \mathbb{R} : y \in S(f), y > L_T(f, x), L_T(f, x) \preceq y\} - T\}$. Then $L_T(f, x)$ will remain the same when and only when x moves between a and b . That is, $L_T(f, y) = L_T(f, x)$ for $y \in I$, $I = [a, b], [a, b), (a, b]$ or (a, b) , whether the boundary is closed or open being determined by which one is larger/smaller in the max and min in the definition of a and b , and whether the supremum and infimum are achieved by a single point or only by a sequence of points. As a result, for any $y \in I$, $g(y) = L_T(f, y) - y = L_T(f, x) - y = d - y$ where $d := L_T(f, x) \in \cap_{y \in I} [y, y + T] \subseteq [b, a + T]$. Thus case 2 corresponds to scenario (1) and case 1 corresponds to scenario (2) in Theorem 5.1.

Next we check the combination rule, that is, the sentence below the two scenarios in the theorem. Firstly assume $g(a) \neq T$. Hence either $g(a) < T$ or $g(a) = \infty$. If $g(a) < T$, consider $g(x)$ for $x \in (a - T + g(a), a)$. Notice that $x + T > a + g(a) = L_T(f, a)$. However, $L_T(f, x)$ can not be equal to $L_T(f, a)$, since otherwise by Lemma 4.3 a will not be the left endpoint of a largest interval on which $g(\cdot)$ is linear. Hence $L_T(f, x) \in [x, x + T] \setminus [a, a + T] = [x, a)$. Since x can be arbitrarily close to a , this implies $g(x) \rightarrow 0$ as $x \uparrow a$.

The argument for the possibility $g(a) = \infty$ is similar. For any $x < a$, $L_T(f, x) \in [x, a)$ or $L_T(f, x) = \infty$. The last instance, however, is not possible when $x > a - T$, since otherwise by Lemma 4.3 the interval I will not be the largest interval on which g is ∞ . Thus $L_T(f, x) \in [x, a)$, which then implies that $g(x) \rightarrow 0$ as $x \uparrow a$.

In the same spirit, we can show that if I is open at a , then $g(a) = 0$. Assume it is not the case. Then $g(a) = \infty$ or $0 < g(a) \leq T$. If $g(a) = \infty$ and $g(x)$ is also infinity on (a, b) or $(a, b]$, the maximality of the interval I is violated; if $g(x) = d - x$ for any $x \in I$ and some $d \in [b, a + T]$, then $L_T(f, x) = d \in [a, a + T]$, which contradicts with $L_T(f, a) = g(a) + a = \infty$ according to the definition of local intrinsic location functional. Hence we must have $0 < g(a) \leq T$. Consider a point $s \in (a, \min(a + g(a), b))$. $L_T(f, s) = d \in [b, a + T] \subseteq [s, a + T]$. However $L_T(f, a) = a + g(a) \in [s, a + T]$, thus $L_T(f, s) = L_T(f, a)$, contradicting with the openness of I on a . Therefore both of the two possibilities fail and $g(a)$ must take value 0.

Now let us turn to the other direction of the proof. The measurability and shift invariance are already given. The value range $L_T(f, a) \in [a, a + T] \cup \{\infty\}$ for any $f \in H$ and $a \in \mathbb{R}$ is easy to check. It remains to show condition (4) in Definition 3.1. Before we proceed, notice that the combination rule leads to the following fact:

Lemma 5.2. *Let $g : \mathbb{R} \rightarrow \mathbb{R} \cup \{\infty\}$ be a function satisfying the combination rule. Then for $x, y \in \mathbb{R}, x < y$ satisfying $g(x) \neq \infty$ and $g(y) \neq \infty$, $g(x) - g(y) \leq y - x$. The equality holds if and only if x and y are in the same maximal interval in Theorem 5.1. Equivalently, let $L_T(t) = g(t) + t$ for $t \in \mathbb{R}$, then for $x, y \in \mathbb{R}, x < y$ satisfying $L_T(x) \neq \infty$ and $L_T(y) \neq \infty$, $L_T(x) \leq L_T(y)$. The equality holds if and only if x and y are in the same maximal interval in Theorem 5.1.*

The proof of this lemma is easy and omitted here.

Let $y_1 < y_2$ be two arbitrary points on the real line. We can assume that $y_2 - y_1 \leq T$, since otherwise the condition $L_T(f, y_2) \in [y_1, y_1 + T]$ can never be satisfied. There are two cases. Case 1: y_1 and y_2 are in the same interval I , on which $g(x) = d - x$ or $g(x) = \infty$. Clearly, in this case

$L_T(f, y_1) = L_T(f, y_2)$. Case 2: y_1 and y_2 are not in the same interval. Say, $y_2 \in I_2$ and $y_1 \notin I_2$, where $I_2 = [a_2, b_2], [a_2, b_2), (a_2, b_2]$ or (a_2, b_2) is the largest interval containing y_2 on which $g(x) = d - x$ or $g(x) = \infty$. Notice that $L_T(f, y_1) \neq L_T(f, y_2)$, since otherwise the monotonicity implies that $L_T(f, x) = L_T(f, y_2)$ for all $x \in [y_1, y_2]$, contradicting with the assumption that I is the largest interval. Our goal is therefore to prove that in this case, $L_T(f, y_2) \in [y_1, y_1 + T] \cap [y_2, y_2 + T] = [y_2, y_1 + T]$ implies $L_T(f, y_1) \in [y_1, y_2]$.

Assume that $L_T(f, y_2) \in [y_2, y_1 + T]$. Firstly, $L_T(f, y_1)$ can not be infinity. Otherwise, let I_1 be the largest interval containing y_1 on which the location takes value ∞ . By the combination rule $\lim_{y \downarrow b_1} g(y) = T$, where b_1 is the right endpoint of I_1 . $b_1 \geq y_1$ so $y_2 - b_1 \leq y_2 - y_1$. Meanwhile $L_T(f, y_2) \in [y_2, y_1 + T]$ implies that $g(y_2) = L_T(f, y_2) - y_2 \leq y_1 + T - y_2$, thus $\lim_{y \downarrow b_1} g(y) - g(y_2) = T - g(y_2) \geq y_2 - y_1$. If equality actually holds for both this inequality and the previous one, then $y_1 = b_1$, and $\lim_{y \downarrow b_1} g(y) - g(y_2) = y_2 - y_1$, hence also $\lim_{y \downarrow y_1} g(y) - g(y_2) = y_2 - y_1$. By Lemma 5.2, $y_1 \geq a_2$, where a_2 is the left endpoint of the maximal interval I_2 containing y_2 . Since $y_1 \notin I_2$, $y_1 = a_2$ and I_2 is open at y_1 . However, by combination rule, this implies that $g(y_1) = T \neq \infty$, contradiction. Thus the two inequalities can not be equalities at the same time. As a result, $\lim_{y \downarrow b_1} g(y) - g(y_2) > y_2 - b_1$, which, however, contradicts with Lemma 5.2. Thus $L_T(f, y_1) \neq \infty$.

Next, notice that $\lim_{y \downarrow a_2} L_T(f, y) = L_T(f, y_2) \in [y_2, y_1 + T]$. If $g(a_2) = T$, then $g(a_2) - \lim_{g \downarrow a_2} g(y) \geq 0 = \lim_{y \downarrow a_2} -a_2$. According to Lemma 5.2, this can only happen if $a_2 \in I_2$. However, $y_1 \leq a_2 \leq L_T(f, a_2) \leq y_1 + T$ and $T = g(a_2) = L_T(f, a_2) - a_2$ implies that $y_1 = a_2$. Together we have $y_1 \in I_2$, contradiction. Thus $g(a_2) \neq T$. Therefore by combination rule, $\lim_{y \uparrow a_2} g(y) = 0$. If $a_2 < y_2$, then by the monotonicity of $L_T(f, \cdot)$ given by Lemma 5.2, $L_T(f, y_1) \leq \lim_{y \uparrow a_2} L_T(f, y) = a_2 \in [y_1, y_2]$. Therefore we only need to consider the case where $a_2 = y_2$. Suppose that in this case $L_T(f, y_1) \geq a_2 = y_2$. By the monotonicity of $L_T(f, \cdot)$ and the fact that $\lim_{y \uparrow a_2} L_T(f, y) = a_2 = y_2$, b_1 must be equal to y_2 , where b_1 is the right endpoint of the maximal interval I_1 containing y_1 , and the inequality above can only be an equality. As a result, I_1 is open at y_2 , therefore

$g(y_2) = T$ by combination rule. This contradicts with the assumption that $L_T(f, y_2) = g(y_2) + y_2 \in [y_2, y_1 + T]$. To conclude, $L_T(f, y_1) < y_2$, hence $L_T(f, y_1) \in [y_1, y_2)$. The second direction of Theorem 5.1 is therefore proved. \square

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